



## On the eigenvalue-eigenvector method for solution of the stationary discrete matrix Riccati equation

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(P) Assume that  $(A, b)$  is reachable, and find a control sequence  $\{u_k\}$  that transfers the initial state  $x_1 = b$  in the system (1) to the origin in a minimal number of steps.

As an immediate consequence of the reachability condition of  $(A, b)$  and of the Cayley-Hamilton theorem we obtain the following.

**Proposition 1:** The time-optimal-control problem (P) has a unique solution with the following properties.

a) The minimal time is  $l = n$  (that is,  $x_{n+1}$  is the first state that can be zeroed).

b) The unique control sequence that solves (P) is given by  $u_i = \alpha_i$ ,  $i = 1, \dots, n$ .

c) The optimal state sequence  $\{x_k\}_{k=1}^n$  obtained by the controls of b) as

$$\begin{cases} x_1 = b \\ x_{k+1} = Ax_k + b\alpha_k; & k = 1, \dots, n-1 \end{cases} \quad (2)$$

forms a basis for  $K^n$  and is the unique state sequence through which  $x_1 = b$  can be steered to the origin in  $n$  steps.

We consider now the problem (P) for a "feedback associate" of system (1), that is, for the system

$$x_{k+1} = \hat{A}x_k + bv_k; \quad (k = 1, 2, \dots) \quad (3)$$

where  $\hat{A} = A + bf$  with  $f$  being a  $1 \times n$   $K$ -matrix. The reachability of (1) obviously implies the reachability of (3) for every  $f$ .

Suppose first that  $f$  is fixed and apply Proposition 1 to the system (3). Part b) of the proposition implies that the optimal control sequence  $\{v_i\}_{i=1}^n$  is given by  $v_i = \beta_i$  where the  $\beta_i$  are the coefficients of the characteristic polynomial  $\psi_{\hat{A}}(\lambda)$  of  $\hat{A}$ :

$$\psi_{\hat{A}}(\lambda) = \lambda^n + \beta_1\lambda^{n-1} + \dots + \beta_{n-1}\lambda + \beta_n.$$

The trajectories of the systems (1) and (3) can be equated by relating their controls through

$$u_k = v_k + fx_k; \quad (k = 1, 2, 3, \dots). \quad (4)$$

Thus, the (unique) minimizing state sequence  $\{x_k\}$  through which  $x_1 = b$  can be driven to the origin in  $n$  steps is the same whether we employ the system (1) or (3). In other words, the state sequence in (2) is a "feedback invariant," that is, it is the same for every feedback associate of (1). From this latter fact and from (4), it follows that the coefficients  $\beta_i$  are related to the  $\alpha_i$  through  $f$  by

$$\beta_i = \alpha_i + fx_i; \quad (i = 1, 2, \dots, n). \quad (5)$$

We turn now to the converse problem.

**Proof of Theorem 1:** Let  $\phi(\lambda) = \lambda^n + \beta_1\lambda^{n-1} + \dots + \beta_{n-1}\lambda + \beta_n$  be any polynomial of degree  $n$  with coefficients in  $K$ . We wish to find  $f$  such that  $\hat{A} = A + bf$  has  $\phi(\lambda)$  as its characteristic polynomial. If such an  $\hat{A}$  exists, then the state sequence  $\{x_k\}_{k=1}^n$  of (2) is optimal also for the pair  $(\hat{A}, b)$ . Moreover, Proposition 1 [applied to the system (3)] implies that the sequence  $v_i = \beta_i$ ,  $i = 1, 2, \dots, n$  must be the unique minimizing control sequence of problem (P) so that, as before, (5) must hold. That  $\hat{A}$  indeed exists as required follows then from the fact that the optimal state sequence  $\{x_i\}$  of (2) forms a basis for  $K^n$  (part c) of Proposition 1) so that (5) has a unique solution  $f$  for every set  $\{\beta_i\}$ .  $\square$

From the above proof of the pole shifting theorem, it is apparent that the theorem can be regarded as a consequence of the uniqueness of the solution to the problem (P) and the "feedback invariance" of the state sequence (2). Also, a crucial fact on which the pole shifting theorem hinges is that the sequence (2) forms a basis for  $K^n$ . A similar point of view was also taken in a recent note by Hautus [5] where the so-called "Heymann Lemma," which extends the pole shifting theorem to multi-input reachable systems, is reproved.

The preceding discussion applies also when the reachability of  $(A, b)$  is not satisfied. In that case, let  $\phi_{(A,b)}(\lambda)$  be the minimal polynomial of  $b$  (relative to  $A$ ) (see, e.g., [6, p. 176]). Then  $\phi_{(A,b)}(\lambda)$  is a factor of the

characteristic polynomial  $\psi_A(\lambda)$  of  $A$  and we can write  $\psi_A(\lambda) = \xi(\lambda) \cdot \phi_{(A,b)}(\lambda)$  for some polynomial  $\xi(\lambda)$ . It is then easily verified that  $\xi(\lambda)$  is invariant under feedback, whereas  $\phi_{(A,b)}(\lambda)$  can be arbitrarily changed by selection of  $f$ .

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## On the Eigenvalue-Eigenvector Method for Solution of the Stationary Discrete Matrix Riccati Equation

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**Abstract**—The purpose of this correspondence is to point out that certain numerical problems encountered in the solution of the stationary discrete matrix Riccati equation by the eigenvalue-eigenvector method of Vaughan [1] can be avoided by a simple reformulation.

The positive definite solution matrix  $P$  of the stationary discrete matrix Riccati equation

$$P = \Phi^T [P^{-1} + R]^{-1} \Phi + Q \quad (1)$$

may be, as shown by Vaughan, found from the eigenvectors of the matrix

$$\kappa_c = \begin{bmatrix} \Phi^{-1} & \Phi^{-1}R \\ Q\Phi^{-1} & \Phi^T + Q\Phi^{-1}R \end{bmatrix}. \quad (2)$$

The eigenvalues of  $\kappa_c$  multiply pairwise to 1, and  $\kappa_c$  may be factorized into

$$\kappa_c = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} \Lambda & 0 \\ 0 & \Lambda^{-1} \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}^{-1} \quad (3)$$

where  $\Lambda$  is diagonal with  $|\Lambda_{ii}| > 1$ .

$P$  is subsequently found from the set of eigenvectors corresponding to eigenvalues of magnitude  $> 1$ :

$$P = W_{21}W_{11}^{-1}. \quad (4)$$

If the system matrix  $\Phi$  has eigenvalues close to 0,  $\kappa_c$  will be severely ill-conditioned, and numerical accuracy is in particular lost in the evaluation of the term  $\Phi^T + Q\Phi^{-1}R$  as small elements of  $\Phi^T$  are added to large elements of  $Q\Phi^{-1}R$ .

This difficulty is avoided as follows.

The eigenvectors of the matrix

$$\kappa_c^* = (\kappa_c + I)^{-1}(\kappa_c - I) \quad (5)$$

are identical to those of  $\kappa_c$ , and the eigenvalues of  $\kappa_c^*$  occur pairwise with opposite signs. Those with positive real parts correspond to eigenvalues of  $\kappa_c$  with magnitudes larger than 1.

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From  $\kappa_c^* = (\kappa_c + I)^{-1}(\kappa_c - I) = (\kappa_c + I)^{-1}S^{-1}S(\kappa_c + I)$  and letting  $S = \begin{pmatrix} \Phi & 0 \\ -Q & I \end{pmatrix}$ , we find

$$\kappa_c^* = \begin{pmatrix} I + \Phi & R \\ -Q & I + \Phi^T \end{pmatrix}^{-1} \begin{pmatrix} I - \Phi & R \\ Q & \Phi^T - I \end{pmatrix}. \quad (6)$$

Hence, inversion of  $\Phi$ , matrix multiplications, and addition of matrices with large and small elements are avoided at the expense of a Gaussian elimination.

In a study of fixed bed reactor control in our Department [2], a  $10 \times 10$  system matrix  $\Phi$  with eigenvalues from 0.99 to about  $10^{-6}$  was encountered. On an IBM 1800 with six digit accuracy, all significant digits were lost using the original  $\kappa_c$ , while the present approach yielded a  $P$ -matrix with five correct digits.

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## On the Solution of the Discrete-Time Lyapunov Matrix Equation in Controllable Canonical Form

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**Abstract**—The solution to the discrete-time Lyapunov matrix equation in controllable canonical form is shown to be the inverse of the Schur-Cohn matrix. A simple constructive procedure of Berkhouit, based on the backwards Levinson algorithm, is discussed and an application of the result in stochastic control is mentioned.

## INTRODUCTION

The discrete-time Lyapunov matrix equation

$$P - FPF' = GQG', \quad (1)$$

familiar from the realm of the stability of time-invariant, linear, discrete-time systems [1], [2], also arises naturally in the calculation of the limiting covariance of the state of such a linear system driven by stationary white noise [3]. Recall that: 1) if the pair  $[F, G]$  is completely reachable and  $Q$  is any positive definite matrix, then the existence of a unique positive definite symmetric solution  $P$  to (1) is necessary and sufficient for all eigenvalues of  $F$  to lie inside the unit circle; and 2) given the state space equation  $x_{k+1} = Fx_k + Gu_k$ , where  $u_k$  is a zero-mean stationary white random process with covariance  $Q$ , and  $F$  is stable, then the limiting covariance  $E[x_k x_k']$  is  $F^k P$  for  $k < \infty$ .

The solution of (1) may be defined in terms of  $F$  and  $G$  by an infinite matrix series in case  $F$  is stable or by solving an equation of the form  $Ax = b$  for  $x$  [2]. But often it is more expedient to transform  $F$  and  $G$  to some matrix pair  $[A \ B]$  which is in canonical form via the relation

$$[A \ B] = [TF T^{-1} \ TG] \quad (2)$$

for some nonsingular  $T$ , so that the solution  $(T^{-1}PT^{-1})$  to (1), where  $F, G$  are replaced by  $A, B$ , is evident or well known. Parks [4], Barnett [2], and Anderson [5] established several such connections between certain canonical solutions to the continuous-time version of (1) when illustrating the connection between classical stability tests. Jury [1] has unified this approach.

In this paper we present the solution to (1) when  $F$  and  $G$  are in controllable canonical form. As this is a very common, simple, and useful canonical form, we feel that, quite apart from the theoretical interest, the result could prove to be quite a powerful tool.

As the explicit nature of the solution to the problem is derived, we are able to draw upon several known results to illustrate how and why the solution may be constructed via the backwards Levinson algorithm. While this is not a new result [12], we do provide a unified, and hence clarified, derivation.

## MAIN RESULT

Parks [4] has shown that if  $F$  and  $G$  have the form

$$F_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & -a_n \\ 1 & 0 & \cdots & 0 & -a_{n-1} \\ 0 & 1 & \cdots & 0 & -a_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -a_1 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 1 - a_n^2 \\ a_1 - a_n a_{n-1} \\ a_2 - a_n a_{n-2} \\ \vdots \\ a_{n-1} - a_n a_1 \end{bmatrix} \quad (3)$$

then the solution  $P_1$  of (1) is the Schur-Cohn matrix associated with the polynomial  $a(z) = z^n + a_1 z^{n-1} + \cdots + a_n$  ( $= \det[zI - F]$ ), i.e.,

$$(P_1)_{ij} = \sum_{k=1}^{\min(i,j)} (a_{i-k} a_{j-k} - a_{n-i+k} a_{n-j+k}). \quad (4)$$

Beginning from this result, we first construct the  $T$  matrix of (2) which takes  $[F_2, G_2]$  to  $[F_1, G_1]$ , where

$$F_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}. \quad (5)$$

Then we examine the transformed solution to (1). We use the following result adapted from Anderson [6], but also derived by Kailath [7] in this context of state variable basis transformations.

**Lemma [7]:** The unique solution  $T$  to the matrix equations

$$TF_2 T^{-1} = F_1 \quad TG_2 = G_1 \quad (6)$$

is the Bezoutian matrix associated with the polynomial pair

$$a(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

and

$$b(z) = (a_{n-1} - a_n a_1) z^{n-1} + (a_{n-2} - a_n a_2) z^{n-2} + \cdots + (1 - a_n^2),$$

i.e.,  $T_{ij}$  is the coefficient of  $x^{i-1} y^{j-1}$  in  $(a(x)b(y) - b(x)a(y))/x - y$ .

**Proof:** Consider the rational function  $W(z) = b(z)/a(z)$  with state space realization in controllable canonical form  $W(z) = G_1'(zI - F_1)^{-1}G_2$ . Identifying  $F_1$  with  $F_2'$  and using [6, Theorem 4] establishes the result.

We are now in a position to derive the main result, which relies upon the recognition of the character of the Bezoutian matrix  $T$  above.

**Theorem:** The Bezoutian matrix  $T$  of the lemma is the Schur-Cohn matrix  $P_1$  of (4) with row order reversed.

Consequently, the solution of the discrete-time Lyapunov matrix equation in controllable canonical form

$$P_2 - F_2 P_2 F_2' = G_2 G_2' \quad (7)$$

is the inverse of the Schur-Cohn matrix associated with the polynomial  $a(z) = z^n + a_1 z^{n-1} + \cdots + a_n$ , i.e.,  $P_2 = P_1^{-1}$ .

**Proof:** It is readily seen that the  $ij$  entry of the Bezoutian matrix